Autumn 2021 Optimization & Machine Learning Talk IV

Image Sharpening

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(a) Original.

(b) Sobolev: $\mu = 0.9$.

(c) Sobolev: $\mu = 0.8$.

(d) Sobolev: $\mu = 0.5$ after 75 iterations.

(e) L^2: $\mu = 0.9$.

(f) L^2: $\mu = 0.8$ after 30 iterations.

Image Blurring/Deblurring

Start with an image u_0, which is a function $L^2(\Omega)$, where $\Omega \subset \mathbb{R}^2$. Want an operator $S_t : L^2(\Omega) \to L^2(\Omega)$, such that $S_t u_0 = u(t)$. Also want $S_s S_t u_0 = S_{s+t} u_0$. This is a *semigroup* property, which is characteristic of parabolic PDE. Thus, we want:

$$u(0) = u_0, \quad \frac{du}{dt} = Au, \quad t > 0 \tag{1}$$

where A is the infinitesimal generator of S_t, i.e.
$Au_0 = \lim_{t\to 0} \frac{d}{dt} \frac{S_t u_0 - u_0}{t}$

Parabolic PDE of Blurring

First idea:

$$u(0) = u_0, \quad \frac{du}{dt} = \Delta u \tag{2}$$

but this leads to isotropic blurring. Gradient descent of $\int_\Omega \|\nabla u\|^2$
Idea of Perona and Malik:

$$u(0) = u_0, \quad \frac{du}{dt} = \nabla \cdot \left(c \left(\|\nabla u\| \nabla u \right) \right), \quad t > 0 \tag{3}$$

where the function c is smooth and $\lim_{s \to \infty} c(s) = 0$ and $\lim_{s \to 0} c(s) = 1$. This is anisotropic blurring. Gradient descent of $\int_\Omega g \left(\|\nabla u\|^2 \right)$

Image Sharpening

Osher and Rudin introduced the shock filter:

$$u(0) = u_0, \quad \frac{du}{dt} = -\left|\nabla u\right| \mathcal{L}(u), \quad t > 0 \tag{4}$$

Typical choices for $\mathcal{L}$ are:

$$\mathcal{L}(u) = \frac{\Delta u}{1 + |\Delta u|}, \quad \text{and} \quad \mathcal{L}(u) = \frac{\Delta_\infty u}{1 + \Delta_\infty u} \tag{5}$$

Only conjectured for continuous u_0, the one-dimensional shock filter has a unique solution. Can be combined with anisotropic diffusion to have a well-posed framework. But, there isn't a pure sharpening framework that has well-posedness results in 2D.

Backwards Heat Equation

Ill-posedness. Take Fourier transform of the backwards heat equation $u_t = -u_{xx}$:

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx+k^2 t}\hat{u}_0(k)dk \tag{6}$$

What's the decay of $\hat{u}_0(k)$ as $|k| \to \infty$?
Let's take $u_0 \in H^k(\mathbb{R})$s, then:

$$\infty > \|u_0\|_{H^k} = \left(\int \left(1+|\xi|^2\right)^k |\hat{u}_0(\xi)|^2 \, d\xi\right)^{1/2} \tag{7}$$

Not enough decay! Even $u_0 \in C^\infty(\mathbb{R}) \cap L^2(\mathbb{R})$ is not enough!

When Can You Solve the Backwards Heat Equation?

Recall the Fourier transform of a Gaussian is a Gaussian. Suppose u_0 is a Gaussian. Then, $\hat{u}_0 = \mathcal{O}\left(e^{-Ck^2}\right)$ for some constant C. Then the decay of $|k|$ as $k \to \infty$ is enough to guarantee a unique solution up to a critical time: $0 < t_c < C$.

If, for example, $u_0(x) = sin(x)/x$, then the Fourier transform is the characteristic function from $\chi_{[-1,1]}$ and the backwards heat equation yields a solution for all time $t > 0$, but this is a rather nonphysical example.

But, anyway it fails for any L^2-perturbation. So, we say it is not well posed.

The Derivative and the Metric

Cannot speak of variational derivatives without notion of metric! This may not be entirely clear yet. Suppose we have a functional $F : \Omega \to \mathbb{R}$. Suppose we have two smooth functions ρ and ϕ. The usual notion of variational derivative from the Calculus of Variations is:

$$\int_\Omega \frac{\delta F}{\delta \rho}(x)\phi(x)dx = \lim_{\epsilon \to 0} \frac{F[\rho + \epsilon\phi] - F[\rho]}{\epsilon} \tag{8}$$

Where does the L^2 metric occur in this? In the left-hand side! That is the L^2 notion of inner product:

$$\left\langle \frac{\delta F}{\delta \rho}, \phi \right\rangle_{L^2(\Omega)} := \int_\Omega \frac{\delta F}{\delta \rho}(x)\phi(x)dx \tag{9}$$

Sobolev Norm

Consider now instead the inner product:

$$\langle v, w\rangle_{H^1(\Omega);\lambda} := (1-\lambda)\langle v, w\rangle_{L^2(\Omega)} + \lambda \sum_{|\alpha|\leq k} \langle D^\alpha v, D^\alpha w\rangle_{L^2(\Omega)} \quad (10)$$

and compute for a given functional F:

$$\left\langle \frac{\delta F}{\delta \rho}, \phi \right\rangle_{H^1(\Omega);\lambda} = \lim_{\epsilon\to 0} \frac{F[\rho+\epsilon\phi] - F[\rho]}{\epsilon} \quad (11)$$

Sobolev Gradient Descent

This leads to the PDE (for sharpening):

$$\frac{du}{dt} = \frac{1}{\lambda}\left(I - (I - \lambda\Delta)^{-1}\right) u \tag{12}$$

Which means:

$$\begin{cases} \frac{du}{dt} = \Delta w \\ w - \lambda\Delta w = u \end{cases} \tag{13}$$

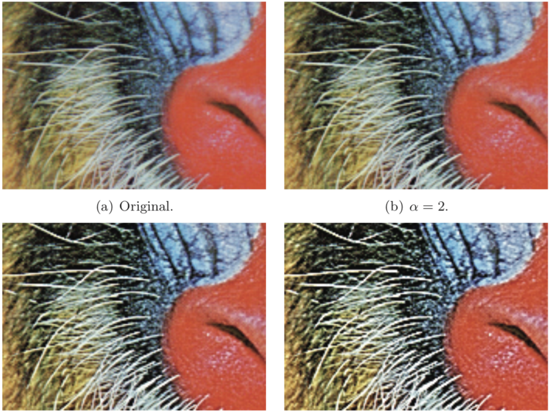
Compactness Property

Why does this work forwards and backwards in time? Well, if $u \in H_0^1(\Omega)$, then so is w. Then, denoting $A := \frac{1}{\lambda}\left(I - (I - \lambda\Delta)^{-1}\right) u$, we compute:

$$\|Au\|_{H_0^1(\Omega)} = \int_\Omega \nabla(Au) \cdot \nabla(Au) dx \tag{14}$$

$$= \int_\Omega \frac{1}{\lambda} \nabla(u - w) \nabla(Au) dx \leq \|u\|_{H_0^1(\Omega)} \|Au\|_{H_0^1(\Omega)} \tag{15}$$

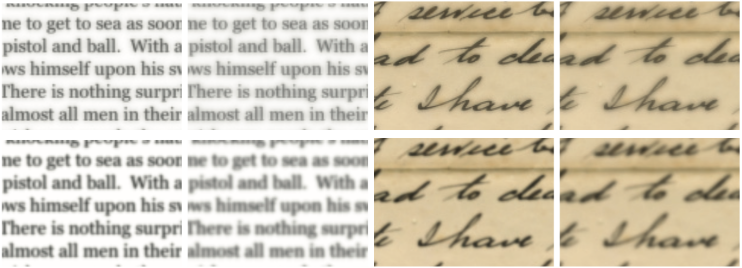
since the Sobolev norm of w is controlled by the Sobolev norm of u. Thus, the operator $u \mapsto Au$ is a bounded linear operator and thus the Sobolev gradient descent is solvable both forward and backwards in time. Note this argument does **not** work for the Laplacian!



(a) Original. (b) $\alpha = 2$.

(c) $\alpha = 5$. (d) $\alpha = 10$.

Figure 10. *Stable Sobolev sharpening without stopping times. For each sharpness factor α the gradient descent PDE* (5.2) *converged to a local minimum in under 25 iterations using a time step $\delta t = 0.1$ (diffusion performed on the luminance component only).*



(a) 75%. (b) 90%. (c) 75%. (d) 90%.

Figure 3. *Sobolev (top) versus L^2 (bottom) diffusion on book and handwriting test images shown when $u \mapsto \int \|\nabla u\|^2$ has decreased by 75% and 90%.*

Questions?

Highlighted Resources

- “Image Sharpening via Sobolev Gradient Flows” J. Calder, A. Mansouri, and A. Yezzi
- “Partial Differential Equations” Lawrence Evans
- “Gradient Flows in Metric Spaces and in the Space of Probability Measures” Luigi Ambrosio and Nicola Gigli and Giuseppe Savaré

Future Talks

Next Talk:

September 30: Brittany Hamfeldt
Topic: Full Waveform Inversion Using the Wasserstein Metric